CALCULATING THE BOUNDARIES OF A SUPERSONIC NONVISCOUS JET ENTERING A SUBMERGED SPACE OR A COMPANION SUPERSONIC FLOW

I. S. Belotserkovets and V. I. Timoshenko

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On the basis of the laws of similitude and the method of linearization, relations are obtained for determining the boundaries of an underexpanded jet in a submerged space or companion hypersonic flow.

In several problems of gasdynamics, particularly in calculating separated flows [1], it is necessary to repeatedly calculate the shape of the boundary of a supersonic, underexpanded jet flowing into a submerged space or a companion supersonic stream. Despite the fact that use of the methods available for obtaining accurate numerical solutions to the equations of gasdynamics [2] for this purpose presents no particular problems, it is more worthwhile to use approximate methods of calculation in such problems [3, 4]. The use of approximate methods substantially reduces the laboriousness of the calculations, as well as the amount of machine time required to perform them. Here, an important role is played by those methods based on the properties of a hypersonic flow in a jet and in an external flow [5].

The similitude property of underexpanded jets [6, 7] can be used to expand the range of application of experimental data and empirical relations. This possibility was illustrated in Part 1 of the present work with the example of the generalization of an empirical relation for the shape of the boundary of a jet in a submerged space [3] in the case of large values of n ($n > 10^3$) and flow into a companion hypersonic stream.

Similitude results can be used only to study the efflux of jets with a high underexpansion n and at hypersonic speeds of the companion flow. In connection with this, in Part 2 we propose an approximate method for calculating the boundary of a jet in the case of low values of n ($n < 10^2$) and at moderate and high supersonic velocities of the external flow.

1. Let us examine the flows of a supersonic jet with characteristic parameters at the nozzle edge into a submerged space or hypersonic companion flow. The index a will designate parameters on the nozzle edge, while ∞ will designate parameters in the external flow. In accordance with the principle of the similitude of flow in underexpanded jets, the shape of the boundary of the jet in a submerged space in the variables $(\bar{y}-1)/R_y$, $\bar{x}/R_x$ [6] is similar. Thus, if $y_0 = f(x_0)$ is the shape of the boundary of the jet in a submerged space with a certain combination of characteristic parameters determining the longitudinal and transverse scales R_x^* and R_y^*, then with any other combination of characteristic parameters with other R_x and R_y, the form of the jet will be determined thus:

$$\bar{y} = 1 + \sqrt{\frac{R_y}{R_y^*}}\left[f\left(\sqrt{\frac{R_x^*}{R_x}}\,\bar{x}\right) - 1\right];\quad \left(\bar{y} = \frac{y}{r_a},\quad \bar{x} = \frac{x}{r_a}\right). \tag{1}$$

Equation (1) can be used to calculate the boundary of a jet in a companion hypersonic flow if we substitute the effective value n_{ef} [7] for the actual value of n. The value of n_{ef} is found from the formula

$$n_{ef} = n\,\frac{1}{\gamma_\infty M_\infty^2 (1-J)}, \tag{2}$$

where

$$n = \frac{P_a}{P_\infty},\quad J = \left(1 + \frac{1}{\gamma_a M_a^2}\right)\left(1 + \frac{2}{\gamma_a - 1}\,\frac{1}{M_a^2}\right)^{-\frac{1}{2}}.$$

As an example of the use of the principle of similitude, let us examine the possibility of expanding the applicability of the relation for a stream boundary proposed in [3] for the case where γ_a, M_a, and n do not fall within the examined range. In this case, the shape of

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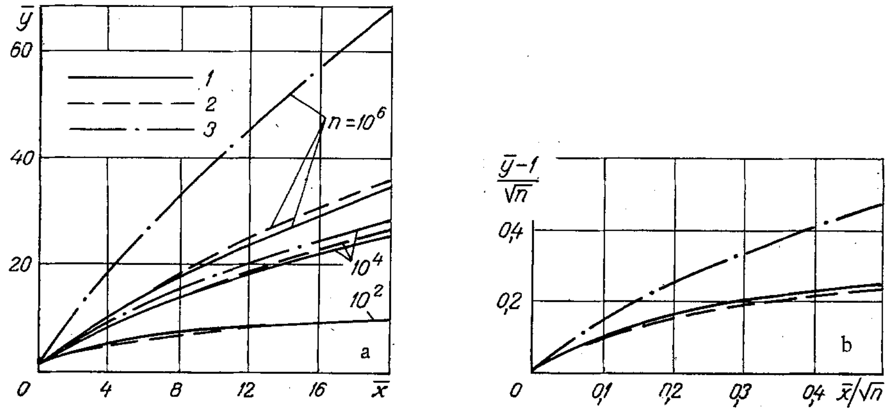

Fig. 1. Comparison of results of calculations of the boundary of a jet by approximate formulas and by the method of characteristics: a: 1) Eq. (3); 2) numerical calculation by the method of characteristics; 3) formula from [3] without allowance for similitude; b: 1) jet boundary in a companion flow according to Eq. (3) at $M_\infty = 10$, $M_a = 4$, $n = 10^4$, $n^* = 10^3$, $\gamma_\infty = 1.4$, $\gamma_a = 1.35$, $\theta_a = 10°$; 2) calculation by the method of characteristics [10]; 3) boundary of jet in a submerged space according to Eq. (3) at $M_a = 4$, $n = 10^4$, $\gamma_a = 1.35$, $\gamma_\infty = 1.4$, $\theta_a = 10°$, $n^* = 10^3$.

the boundary

$$\bar{y} = 1 + \sqrt{\frac{\gamma_a}{\gamma_a - 1}\,\frac{\gamma_a^* - 1}{\gamma_a^*}\,\frac{n}{n^*}}\left[f\left(\frac{M_a^*}{M_a}\sqrt{\frac{\gamma_a^*}{\gamma_a}\,\frac{n^*}{n}}\,\bar{x}\right) - 1\right]. \tag{3}$$

The values of the parameters γ_a^*, M_a^*, and n^* in Eq. (3) were taken from [3], while γ_a, M_a, and n are the values at which it is necessary to determine the shape of the jet in the submerged space. In order to use Eq. (3) in the case of a companion flow, it is necessary to introduce n_{ef} from (2) in place of the actual underexpansion.

As an example of the use of Eq. (3), Fig. 1a shows the boundaries of severely underexpanded jets flowing into a submerged space calculated by Eq. (3) at $n = 10^2$, 10^4, and 10^6. Also shown are the boundaries of jets calculated by [3] without the use of the similitude property and the results of a numerical calculation from [2]. Figure 1b illustrates the possibility of using Eq. (3) to calculate the boundary of a jet in a companion hypersonic flow on the initial section of the first roll.

2. The equation from [3] for the case of flow into a companion stream can be generalized by means of the laws of similitude [6, 7] only at high underexpansions $n > 10^2$–10^3 and with hyperbolic flow into the companion stream.

On the initial section of a nonviscous jet entering a submerged space or a companion supersonic flow at low values of the effective underexpansion $n_{ef} < 10^{12}$ and moderate and large numbers M_∞, the form of the boundary can be calculated using the approach taken in [8]. The essence of this approach is the linearization of the equations of axisymmetric gasdynamics in the neighborhood of the solution for plane flow. Here, it is convenient to use the equations in characteristic form, written in a polar system of coordinates (r, θ) within the fan of rarefaction waves I, II and in a Cartesian system of coordinates (ξ, η) in regions bounded by the characteristic curve of the second family II and the stream boundary, shock wave AB, and the boundary in the companion supersonic flow AC (Fig. 2). The general form of the equations will be:

a) in the region ADE

$$\frac{dr_I}{r_I} = \operatorname{tg}(\theta + \beta + \mu)\, d\theta, \tag{4}$$

$$d[f(\mu) - \beta] = \frac{\sin\mu \sin\beta}{\cos(\theta + \beta + \mu)}\,\frac{r_I\, d\theta}{r_a + r_I \cos\theta};$$

b) in the regions AEC and BAC

$$d\eta_{I,II} = \operatorname{tg}(\delta \pm \mu)\, d\xi, \tag{5}$$

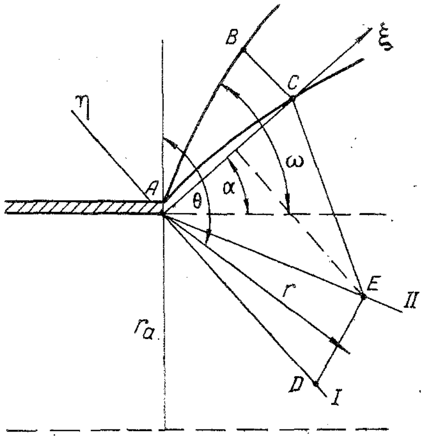

Fig. 2. Flow pattern in vicinity of nozzle edge.

$$d\,[f(\mu) \mp \beta] = \frac{\sin\mu \sin\beta}{\cos(\delta \pm \mu)}\;\frac{d\xi}{r_a + \xi \sin\alpha + \eta_{I,II}\cos\alpha}.$$

Here the top sign pertains to the characteristic curves of family I; the lower sign pertains to the characteristic curves of family II; $f(\mu) = (1/\lambda)\operatorname{arctg}\lambda \operatorname{ctg}\mu + \mu$; $\lambda = \sqrt{(\gamma - 1)/(\gamma + 1)}$ and γ is the ratio of the specific heats in the jet (region AEC) or in the external flow region (region BAC); α is the angle of inclination of the axis ξ to the symmetry axis; $\delta = \beta - \mu$.

In Eqs. (4), (5), we assume that the functions which come after the differentials of μ, θ, β, and δ are known and are equal to the corresponding values in plane flow, i.e., are determined from the Prandtl–Maier solution. Thus, the effect of axial symmetry on the flow is accounted for by the factor $r_I/(r_a + r_I \cos\theta)$ in case a) and $(r_a + \xi \sin\alpha + \eta_{I,II}\cos\alpha)^{-1}$ in case b).

Performing the appropriate integration along the characteristic curves of families I (DE) and (EC) and II (BC), we obtain the following relations for determining the complexes $[f(\mu) - \beta]$ and $[f(\mu) + \beta]$ at point C of the stream boundary:

$$[f(\mu) - \beta]_C = [f(\mu) - \beta]_D - \frac{\psi}{2}\int_{\theta_1}^{\theta_2} \frac{(\cos\theta - \lambda \operatorname{ctg} t \sin\theta)\,d\theta}{r_a \sqrt{|\sin t|\,|\cos t|^{1/\lambda^2}} + \psi\cos\theta} + \frac{1}{\operatorname{ctg}\mu_{AE} + \operatorname{ctg}\alpha}\,\mathrm{Ln}\,\frac{r_a + \xi\sin\alpha}{r_a + \frac{1}{2}\xi(\sin\alpha - \operatorname{tg}\mu_{AE}\cos\alpha)}; \tag{6}$$

$$[f(\mu) + \beta]_C = [f(\mu) + \beta]_B + \frac{1}{\operatorname{ctg}\mu_{AB} - \operatorname{ctg}\alpha}\,\mathrm{Ln}\,\frac{r_a + \sin\alpha\xi}{r_a + m\sin\alpha\xi}, \tag{7}$$

where

$$\psi = \frac{1}{2}\,\frac{\xi}{\cos\mu_{AE}}\sqrt{|\sin t_2|\,|\cos t_2|^{1/\lambda^2}};\quad \theta_1 = \frac{\pi}{2} - \beta_{AD} - \mu_{AD};$$

$$\theta_2 = \frac{\pi}{2} - \beta_{AE} - \mu_{AE};\quad t = \lambda(\theta + c);$$

$$c = -f(\mu_{AD}) + \beta_{AD} - \frac{\pi}{2};\quad m = \frac{1 + \operatorname{ctg}\alpha \operatorname{tg}(\omega - \alpha)}{1 + \operatorname{ctg}\mu_{AB}\operatorname{tg}(\omega - \alpha)}.$$

Equations (6) and (7) constitute a system of two equations relative to $\beta = \operatorname{arctg}(dy/dx)$, M_f, M_c, the angle of inclination, and the Mach numbers above and below the stream boundary. Considering the equality of the pressures P_f and P_c on the line $y = y(x)$ with isentropic flow along the line, we obtain an equation connecting M_f and M_c at point C:

$$\frac{\left(1 + \frac{\gamma_\infty - 1}{2}M_\infty^2\right)^{\frac{\gamma_\infty}{\gamma_\infty - 1}}}{\left(1 + \frac{\gamma_a - 1}{2}M_a^2\right)^{\frac{\gamma_a}{\gamma_a - 1}}} = k, \tag{8}$$

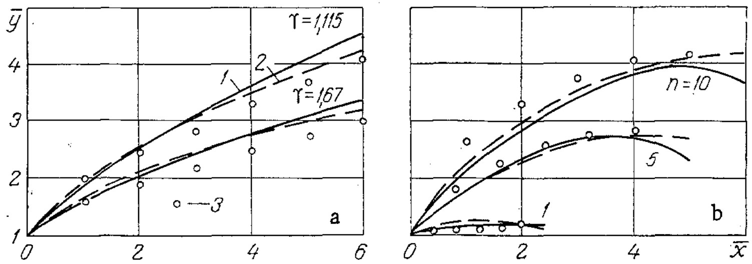

Fig. 3. Form of the boundary of a jet flowing into a submerged space at a low underexpansion: 1) calculation by the proposed method; 2) calculation by the method of characteristics; 3) calculation by the method of Cherkez [9, p. 369]; a) M_a = 1.5, γ_a = 1.115, γ_∞ = 1.4, n = 1, 5, 10; b) M_a = 3, γ_a = 1.67, 1.115, n = 10.

$k = P_{0f}/P_{0c}$; P_{0f}, stagnation pressure in the external supersonic flow behind the shock wave; P_{0c}, stagnation pressure in the stream.

Solving the system of equations (6)-(8), we obtain a transcendental equation which should be satisfied by the Mach number M_c at an arbitrary point C on the stream boundary:

$$\Phi(M_c) = [f(\mu_c)]_C + [f(\mu_f)]_C - [f(\mu_c) - \beta_c]_D + [f(\mu_f) + \beta_f]_B + \frac{\psi}{2}\int_{\theta_1}^{\theta_2} \frac{(\cos\theta - \lambda\,\mathrm{ctg}\,t\sin\theta)\,d\theta}{r_a - \sqrt{|\sin t||\cos t|^{1/\lambda^2} + \psi\cos\theta}} -$$

$$-\frac{1}{\mathrm{ctg}\,\mu_{AE} + \mathrm{ctg}\,\alpha}\,\mathrm{Ln}\,\frac{r_a + \xi\sin\alpha}{r_a + \frac{\xi}{2}(\sin\alpha - \mathrm{tg}\,\mu_{AE}\cos\alpha)} - \frac{1}{\mathrm{ctg}\,\mu_{AB} - \mathrm{ctg}\,\alpha}\,\mathrm{Ln}\,\frac{r_a + \sin\alpha\xi}{r_a + m\sin\alpha\xi} = 0. \qquad (9)$$

The method of secants is used to find the root of (9). The next approximation of the root $M_{c_{n+1}}$ is found from the formula

$$M_{c_{n+1}} = M_{c_n} - \frac{(M_{c_n} - M_{c_{n-1}})\,\Phi(M_{c_n})}{\Phi(M_{c_n}) - \Phi(M_{c_{n-1}})}. \qquad (10)$$

We need to assign M_{c_0} and M_{c_1} for the beginning of the process. For M_{c_0}, we take the solution for plane flow at the point A, while $M_{c_1} = M_{c_0}(1 + \varepsilon)$, where ε is a certain small number; $\varepsilon \sim 10^{-2}$.

It is sufficient to use Eq. (6) to determine β = arctg (dy/dx) in the case of flow of a jet into a submerged space, since here M_c = const.

The process of constructing the form of the boundary of a jet flowing into a companion supersonic flow or into a submerged space may be represented in the simplest form as follows. We assign on the symmetry axis of the flow the sequence of points $x_{i+1} = x_i + \Delta x$, $x_0 = 0$, $i = 0, 1, 2, \ldots$. We approximate the form of the stream boundary on the interval $[x_i, x_{i+1}]$ by a segment of a straight line originating from the point (x_i, y_i) and inclined at an angle β_i to the symmetry axis. We determine the value of β_{i+1} at point C with the coordinates (x_{i+1}, y_{i+1}) from (6) under the condition that $\xi_{i+1} = \sqrt{y_{i+1}^2 + x_{i+1}^2}$ and $y_{i+1} = y_i + \mathrm{tg}\beta_i\Delta x$. Knowing the value of β_{i+1} at point C, we can continue the construction of the stream boundary by an amount Δx on the interval $[x_{i+1}, x_{i+2}]$, etc.

To check the above, we performed calculations and compared the results with the results obtained by numerical methods. Figure 3 shows the results obtained by the present method (solid lines) and the method of characteristics (dashed lines) for the boundary of a stream flowing into a submerged space. Figure 4 compares the results obtained by the present method (solid curves) with accurate numerical results (dashed curves) taken from [10, 11] for the boundary of a supersonic nonviscous stream flowing into a compansion supersonic flow.

The above comparison, as well as comparison of our results with the results obtained by other approximate methods, shows that the method proposed here provides satisfactory accuracy on the initial section of the first roll at moderate values of underexpansion.

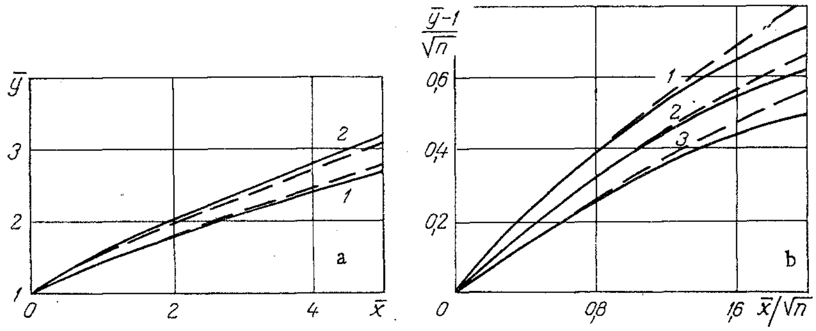

Fig. 4. Form of boundary of a jet flowing into a companion flow: a) $M_\infty = 5$, $M_a = 3$, $n = 62.9$, $\theta_a = 10°$, $\gamma_\infty = 1.4$, 1) $\gamma_a = 1.4$, 2) $\gamma_a = 1.15$; b) $M_a = 4$, $n = 10^2$, $\theta_a = 10°$, $\gamma_a = 1.4$, $\gamma_\infty = 1.4$, 1) $M_\infty = 3$, 2) $M_\infty = 6$, 3) $M_\infty = 10$.

NOTATION

n, underexpansion; M, Mach number; γ, ratio of specific heats; P, static pressure in the flow; θ_a, angle of taper of nozzle; R_x, R_y, longitudinal and transverse scales; r_a, radius of nozzle; n_{ef}, effective underexpansion; μ, Mach angle; β, angle of inclination of velocity vector to symmetry axis of flow; k, ratio of stagnation pressures of flows; M_{cn}, approximations of Mach number at point C on the stream boundary.

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